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S_g- Continuity in Topological Spaces

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Abstract

The aim of the present paper, is to introduce and study a new class of functions which is strictly placed between the class of S_c-continuous functions and the class of semi-continuous function, we call it the class of S_g-continuous functions. The basic properties, characterizations and relationships with some other classes of continuity are also obtained.

Key Words:
S_g-open set,
Semi-open set,
g-closed set,
S_g-continuous Function

1. Introduction and Preliminaries

Throughout the present paper (X, τ) and (Y, σ) or simply X and Y denote topological spaces on which no separation axioms are assumed unless explicitly stated. We recall the following definitions, notations and terminology. The closure (resp. interior) of a subset A of X is denoted by clA (resp. $int\,A$). A subset A of a space X is said to be semi-open [17] (resp., preopen [20], α-open [24], β-open [1], regular open [32], δ-semi-open [28], and g-closed [16]) set, if $A \subseteq clintA$ (resp. $A \subseteq intclA$, $A \subseteq intclintA$, $A \subseteq clintclA$, $A = intclA$, $A \subseteq clint_{\delta}A$ and $clA \subseteq U$ whenever U is open and $A \subseteq U$). The complement of a semi-open (resp. preopen, α-open, β-open, regular open, δ-semi-open and g-closed) set is said to be semi-closed (resp. preclosed, α-closed, β-closed, regular closed, δ-semi-closed and g-open).The family of all semi-open (resp., preopen, α-open, β-open, regular open, δ-semi-open and g-closed) subsets of a topological space X is denoted by $SO(X)$ (resp., $PO(X), \alpha O(X), \beta O(X), RO(X), \delta SO(X)$ and $GC(X)$). A subset A of a space X is called θ-semi-open [11] (resp., δ-open [34], θ-open [34]) if for each $x \in A$, there exists a semi-open (resp., open, open) set G such that $x \in G \subseteq clG \subseteq A$ (resp., $x \in G \subseteq intclG \subseteq A, x \in G \subseteq clG \subseteq A$). The complement of θ-semi-open (resp., δ-open and θ-open) set is said to be θ-semi-closed (resp. δ-closed, θ-closed). The family of all θ-semi-open (resp., δ-open and θ-open) subsets of a topological space X is denoted by $\theta SO(X)$ (resp. $\delta O(X)$ and $\theta O(X)$). A subset A of a space X is called S_c-open[13] (resp. S_P-open[30], S_S-open[15], S_β-open[12] and S_g-open[3]) if for each $x \in A \in SO(X)$, there exists a closed (pre-closed, semi-closed, β-closed, g-closed) set F such that $x \in F \subseteq A$. Further, a δ-semi-open A of a space X is said to be δs_C-open[2], if for each $x \in A$ there exists a closed set F such that $x \in F \subseteq A$. The family of all S_c-open (resp. S_P-open, S_S-open, S_β-open, S_g-open and δs_C-open) subsets of a topological space X is denoted by $S_cO(X)$ (resp. $S_pO(X), S_sO(X), S_\beta O(X)$, $S_gO(X)$ and $\delta s_CO(X)$). A function $f: X \to Y$ is said to be θs–continuous [14] (resp. continuous [31], α-continuous [21], super continuous [23], totally continuous [10], perfectly continuous [25], totally-semi-continuous [27], RC-continuous [6], contra-continuous [5], precontinuous [19]) if the inverse image of each open subset of Y is θ-semi-open (resp. open, α-open, δ-open, clopen, clopen, semi-clopen, regular closed, closed, preopen) in X. A function $f: X \to Y$ is said to be slightly continuous [10] (resp. semi-continuous [17], clopen continuous [29], contra-α-continuous [9], contra pre-continuous [8], β-continuous [1], S_c-continuous [13], S_p-continuous [30], S_s-continuous [15], S_β-continuous [12] and δs_c-continuous [2])) if the inverse image of each clopen (resp. open, open, open, open, open, open, open, open, open and open) subset of Y is open (resp. semi-open, clopen, α-closed, pre-closed, β-open and S_c-open) in X. A function $f: X \to Y$ is said to be quasi-θ-continuous[26] if for each $x \in X$ and each θ-open set V of Y containing *f(x)*, there exists a θ-open set U of X containing x such that $f(U) \subseteq V$. A function $f: X \to Y$ is said to be R-map [22] if the inverse image of each regular open subset of Y is regular open in X.

Definition 1.1 A semi-closed set A of a space X is called S_g-closed set if $X \backslash A$ is S_g-open set. The family of all S_g-closed subsets of a space X is denoted by $S_gC(X)$.

Proposition 1.2 [3] A subset A of a space X is S_g-open if and only if A is semi-open and A is a union of g-closed sets.

Proposition 1.3 A subset A of a space X is S_g-closed if and only if A is semi-closed set and it is an intersection of g-open sets.

Proof. Obvious.

The intersection of all S_g-closed sets of X containing A is called the S_g-closure of A and is denoted by $S_gcl(A)$. The union of all S_g-open sets of X containing A is called the S_g- interior of A and is denoted by $S_gint(A)$.

Definition 1.4 A space X is said to be:

1- extermally disconnected [35] if the closure of every open set in X is open .

2- locally indiscrete [4] if every open subset of X is closed.

3- $T_{\frac{1}{2}}$ - space [16] if every g-closed set is closed.

Proposition 1.5 [3] 1-Every θ-open (resp.θ-semi open) set of X is S_g-open.

2- Every regular closed set of a space X is S_g-open.

3- Every δs_C-open and S_c–open set of a space X is S_g-open.

Proposition 1.6 [3] If X is a locally indiscrete space, then $SO(X) = S_gO(X)$.

Theorem 1.7 The following conditions are equivalent for a space (X, T):

1- X is extremally disconnected

2- Every semi-open subset of X is preopen [33]

3- Every semi-open subset of X is α–open [18]

Proposition 1.8 [3] If X is a $T_{\frac{1}{2}}$- space, then $S_gO(X) = S_cO(X)$.

Theorem 1.9 The following conditions are equivalent for the extremally disconnected space (X, τ):

1-$RO(X) = RC(X)$ [7]

2- $\delta O(X) = \theta SO(X)$ [36]

Corollary 1.10 For any subset A of a space X, $X \backslash S_gint(A) = S_gcl(X \backslash A)$.

Proof. Let $x \in X\backslash S_g int(A)$. Then $x \notin S_g int(A)$. Therefore, for each S_g-open set U containing x, $x \in U \subseteq S_g cl(U) \nsubseteq A$, so that $S_g cl(U) \cap (X\backslash A) \neq \emptyset$ for every S_g-open set U containing x. Hence $x \in S_g cl(X\backslash A)$. Hence, $X\backslash S_g int(A) = S_g cl(X\backslash A)$.

Proposition 1.11 [3] If X is a T_1-space, then $SO(X) = S_c O(X) = S_g O(X)$.

Proposition 1.12 [3] For any subset A of a space X, the following conditions are equivalent:

1- A is regular closed.
2- A is closed and S_g-open.
3- A is closed and semi-open
4- A is closed and β-open.
5-A is α-closed and β-open.
6- A is pre-closed and β-open.

Corollary 1.13 [3] Let Y be a clopen subspace of a space X. If $A \in S_g O(X)$, then $A \cap Y \in S_g O(Y)$.

Proposition 1.14 [3] Let A be a subset of a subspace Y of a space X. If $A \in S_g O(Y)$ and $Y \in RC(X)$, then $A \in S_g O\ (X)$.

Proposition 1.15 [3] Let A and B be two subsets of a space X. If $A \in S_g O(X)$ and B is clopen, then $A \cap B \in S_g O(X)$.

Proposition 1.16 [3] If $A \in S_g O(X)$ and $B \in S_g O(Y)$, then $A \times B \in S_g O(X \times Y)$.

Corollary 1.17 [3] If X is a $T_{\frac{1}{2}}$ - space, then:

1- $S_g O(X) \subseteq S_s O(X)$.
2-$S_g O(X) \subseteq S_\beta O(X)$.
3-$S_g O(X) \subseteq S_p O(X)$.

2. S_g-continuous Functions:

Definition 2.1 A function $f: (X, \tau) \to (Y, \sigma)$ is called ***S_g*-continuous** at a point $x \in X$, if for each open set U of Y containing $f(x)$ there exists an S_g-open set G in X containing x such that $f(G) \subseteq U$. If f is S_g-continuous at every point x of X, then it is called S_g-continuous.

Proposition 2.2 A function $f: (X, T) \to (Y, \sigma)$ is S_g-cotinuous if and only if for every open subset U of Y, $f^{-1}(U)$ is an S_g-open in X.

Proof. Let f be an S_g-continuous function and O be any open set in Y. To show that $f^{-1}(O)$ is an S_g-open set in X, if $f^{-1}(O) = \emptyset$, implies that $f^{-1}(O)$ is an S_g-open set in X, and if $f^{-1}(O) \neq \emptyset$, then for each $x \in f^{-1}(O)$, we have $f(x) \in O$. Since f is S_g-continuous, then there exists an S_g-open set U_x in X such that $x \in U_x$ and $f\ (U_x) \subseteq O$. That is, $x \in U_x \subseteq f^{-1}(O)$. This shows that $f^{-1}(O)$ is an S_g-open in X.

Conversely, let $x \in X$ and let V be an open set in Y containing $f(x)$. Then $x \in f^{-1}(V)$ and by hypothesis $f^{-1}(V)$ is S_g-open in X containing x, so $f\ (f^{-1}(V))) \subseteq V$. Therefore, f is S_g-continuous.

Proposition 2.3 Every S_g-continuous function is semi-continuous.

The converse of Proposition2.3 is not true in general as shown in the following example:

Example 2.4 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $SO(X, \tau) = \{\emptyset, X, \{b\}, \{b, c\}, \{a, b\}\}$, $S_g O(X, \tau) = \{\emptyset, X, \{a, b\}, \{a, c\}\}$, $SO(X, \sigma) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $S_g O(X, \sigma) = \{\emptyset, X, \{a, c\}, \{b, c\}\}$. If the function $f: (X, \tau) \to (X, \sigma)$ is defined by $f(a) = f(b) = b$ and $f(c) = c$, then f is semi-continuous but not S_g-continuous.

Proposition 2.5 Every θ_s-continuous function is S_g-continuous.

The converse of Proposition 2.5 not true in general as shown in the following example:

Example 2.6 $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ $SO(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, $S_gO(X) = \{\emptyset, X, \{a, c\}\}$ and $\theta SO(X) = \{\emptyset, X\}$ if the function $f: X \to X$ defined by $f(a) = f(c) = a$ and $f(b) = b$, hence f is S_g-continuous since $\{a\} \in \tau$ and $\{a, c\} \in S_gO(X)$ but $\{a, c\} \notin \theta SO(X)$.

Proposition 2.7 Every S_c-continuous and δs_c-continuous function is S_g-continuous.

The converse of Proposition2.7 is not true in general as shown in the following example:

Example 2.8 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $SO(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $GC(X) = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$. Hence $S_gO(X) = \{\emptyset, \{a, c\}, \{a, b\}, X\}$ and $S_cO(X) = \delta s_cO(X) = \{\emptyset, X\}$. If the function $f: X \to X$ defined by $f(a) = f(c) = a$ and $f(b) = b$, then f is S_g-continuous. Since $\{a\} \in \tau$ and $\{a, c\} \in S_gO(X)$ but neither S_c -continuous nor δs_C -continuous $\{a, c\} \notin S_cO(X) = \delta s_CO(X)$.

The following examples show that the S_g-continuous and quasi-θ-continous function are independent concepts:

Example 2.9 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. If $f: (\mathcal{R}, T_{cof}) \to (X, \tau)$ is a function given by

$$f(x) = \begin{cases} c & \text{if} \quad x \notin \{0,1\} \\ a & \text{if} \quad x = 0 \\ b & \text{if} \quad x = 1 \end{cases}.$$

Since the only open set which contains $f(x)$ is X for each $x \in \mathcal{R}$, then $\mathcal{R}$ is θ-open which contain the point x and $f(R) = X \subseteq X$. So f is quasi-θ-continuous but f is not S_g-continuous function. Since $\{a\}$ is open in X but $f^{-1}(\{a\}) = \{0\}$ which is not S_g-open in $\mathcal{R}$.

Example 2.10 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, x, \{a, c\}, \{b, d\}\}$ then $SO(X, \tau) = P(X) \backslash \{\{c\}, \{d\}, \{c, d\}\}$, $S_gO(X, \tau) = \{\emptyset, X, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\theta O(X, \tau) = \{\emptyset, X\}$. The identity function $f: (X, \tau) \to (X, \sigma)$ is not quasi-θ-continuous function. Since $\{a, c\}$ is θ-open in (X, σ) , but $f^{-1}(\{a, c\}) = \{a, c\}$ is not θ-open in (X, τ). However, it is easy to see that f is S_g-continuous function.

Proposition 2.11 Every totally continuous (resp. perfectly continuous) function is S_g-continuous.

Proof. Let V be any open subset in Y. Then $f^{-1}(V)$ is clopen in X. Hence $f^{-1}(V) \in S_gO(X)$, then by Proposition2.2, f is S_g-continuous.

Proposition 2.12 A function $f: X \to Y$ is S_g-continuous if and only if f is semi-continuous and for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a g-closed set F of X containing x such that $f(F) \subseteq V$.

Proof. Let $f: X \to Y$ be a S_g-continuous and also let $x \in X$ and V be any open set in Y containing $f(x)$. By hypothesis, there exists an S_g-open set U of X containing x such that $f(U) \subseteq V$. Since U is an S_g-open set and $x \in U$, then there exists a g-closed set F of X such that $x \in F \subseteq U$. Therefore, we have $f(F) \subseteq V$. Since f is S_g-continuous, then by Proposition 2.3, f is semi-continuous.

Conversely, let V be any open set in Y. We have to show that $f^{-1}(V)$ is S_g-open in X. Since f is semi-continuous, then $f^{-1}(V)$ is semi-open in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and by hypothesis, there exists a g-closed set F of X containing x such that $f(F) \subseteq V$, which implies that $x \in F \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is an S_g-open set in X. Hence by Proposition2.2, f is S_g–continuous.

Proposition 2.13 If $f: X \to Y$ is a function and X is locally indiscrete, then f is S_g-continuous if and only if it is semi-continuous.

Proof. Let f be semi-continuous and X be locally indiscrete .Let U be any open subset in Y. Then $f^{-1}(U)$ is semi-open in X. Since X is locally indiscrete, then by Proposition1.6, $f^{-1}(V) \in S_gO(X)$. Thus f is an S_g-continuous function.

Conversely, let f be S_g-continuous. Then by Proposition 2.3, f is semi-continuous.

Proposition 2.14 If $f: X \to Y$ is a function and X is a T_1-space, then the following properties are equivalent:

1- f is S_g-continuous

2- f is semi-continuous
3- f is S_c-continuous.
Proof. (1) → (2) Let $f: X \to Y$ be an S_g-continuous function. Then by Proposition 2.3, f is a semi-continuous function.
(2) → (3) Let $f: X \to Y$ be a semi-continuous function and let V be open in Y. Then $f^{-1}(V)$ is semi-open in X. Since for all $x \in f^{-1}(V)$, $\{x\}$ is closed and $x \in \{x\} \subseteq f^{-1}(V)$, so $f^{-1}(V) \in S_cO(X)$. Thus f is an S_c-continuous function.
(3) → (1) Follows from Proposition2.7.
Proposition 2.15 If $f: X \to Y$ is a totally-semi-continuous function and X is locally indiscrete, then f is S_g-continuous.
Proof. Let f be totally-semi-continuous and X be locally indiscrete. Let V be any open subset of Y. Since f is totally-semi-continuous, then $f^{-1}(V)$ is semi- clopen in X. This implies that, $f^{-1}(V) \in SO(X)$. Since X is locally indiscrete, then by Proposition1.6, $f^{-1}(V) \in S_gO(X)$. Thus f is S_g-continuous.

The following example shows that a slightly continuous function need not be S_g-continuous in general, even on a locally indiscrete space.
Example 2.16 Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}\}$. Then $SO(X, \tau) = S_gO(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\}$. It's clear that (X, τ) is locally indiscrete, so if we define the function $f: (X, \tau) \to (X, \sigma)$ by $f(a) = b$ and $f(b) = f(c) = c$, then f is slightly continuous but f is not S_g-continuous. Since $\{a\}$ is open in (X, σ) but $f^{-1}(\{a\}) = \{b\}$ which is not S_g-open in (X, τ) .
Theorem 2.17 If $f: X \to Y$ is an S_g-continuous function and X is extermally disconnected, then f is α-continuous (resp. precontinuous).
Proof. Let U be any open subset of Y. Since f is S_g-continuous, then $f^{-1}(U) \in S_gO(X)$, whence $f^{-1}(U) \in SO(X)$. Since X is extermally disconnected, then by Theorem1.7, $f^{-1}(U) \in \alpha O(X)$ (resp. $f^{-1}(U) \in PO(X)$). Thus, f is α-continuous (resp. precontinuous).
Theorem 2.18 Let X be an extermally disconnected space and $f: X \to Y$ is a super continuous (resp. R-map) function. Then f is S_g-continuous.
Proof. Let f be super continuous (resp. R-map) and X is extermally disconnected, and let V be any open subset in Y. Then $f^{-1}(V)$ is δ-open (resp. regular open) in X. Since X is extermally disconnected, then by Theorem1.9, $f^{-1}(V) \in \theta SO(X)$ (resp. $f^{-1}(V) \in RC(X)$) and by Proposition1.5, $f^{-1}(V) \in S_gO(X)$. Thus by Propositioin2.2, f is S_g-continuous.
Corollary 2.19 If $f: X \to Y$ is a function and X is a $T_{\frac{1}{2}}$-spsce, then f is S_g-continuous if and only if it is S_C-continuous.
Proof. Follows by Proposition1.8
Proposition 2.20 For a function$f: X \to Y$, the following statements are equivalent:
1- f is S_g-continuous function.
2- $f^{-1}(V)$ is an S_g-closed set in X, for each closed set V in Y.
3- $f(S_gcl(A)) \subseteq cl(f(A))$, for each subset A of X.
4- $S_gcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$, for each subset B of Y.
5- $f^{-1}(intB) \subseteq S_gint(f^{-1}(B))$, for each subset B of Y.
Proof. (1) → (2) Let V be any closed subset of Y. Then $Y \backslash V$ is open in Y, so by Proposition 2.2, $f^{-1}(Y \backslash V)$ $= X \backslash f^{-1}(V)$ is S_g-open in X. Thus $f^{-1}(V)$ is S_g-closed in X.
(2) → (3) Let $A \subseteq X$. Then $f(A) \subseteq Y$. Since $f(A) \subseteq clf(A)$, and by (2) $f^{-1}(clf(A))$ is S_g-closed in X and $A \subseteq f^{-1}(clf(A))$, then $S_gcl(A) \subseteq (clf(A))$.This implies that, $f(S_gcl(A)) \subseteq clf(A)$.
(3) → (4) Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$ and by (3) we have $f(S_gcl(f^{-1}(B))) \subseteq clf(f^{-1}(B)$. Therefore, $f(S_gcl(f^{-1}(B))) \subseteq cl(B)$. This implies that, $S_gcl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(4) → (5) Let $B \subseteq Y$. Then $Y\backslash B \subseteq Y$. Therefore by (4) $S_gcl(f^{-1}(Y\backslash B)) \subseteq f^{-1}(cl(Y\backslash B))$, then $S_gcl(X\backslash f^{-1}(B)) \subseteq f^{-1}(Y\backslash intB)$. This by Corollary 1.10 implies that, $X\backslash S_gint(f^{-1}(B)) \subseteq X\backslash f^{-1}(intB)$. Hence $f^{-1}(intB) \subseteq S_gint(f^{-1}(B))$.
(5) → (1) Let V be any open subset of Y. Then by (5) $f^{-1}(int(V)) \subseteq S_gint(f^{-1}(V))$. This implies that $f^{-1}(V) \subseteq S_gint(f^{-1}(V))$. Hence $f^{-1}(V)$ is an S_g-open set in X. Thus f is an S_g-continuous function.

Proposition 2.21 If $f:(X,T) \to (Y,V)$ is a continuous function and X is a T_1-space, then it is S_g-continuous.

Proof. Since every open set is a semi-open set and X is a T_1-space, so by Proposition1.11, $SO(X) = S_gO(X)$. Therefore, every open set is S_g-open, and hence f is S_g-continuous.

Proposition 2.22 Let $f: X \to Y$ be an S_g-continuous function. If Y is a subspace of Z, then $f: X \to Z$ is S_g-continuous.

Proof. Let U be an open set in Z. Then $U \cap Y$ is an open in Y. Since f is S_g-continuous, by Proposition 2.2, $f^{-1}(U \cap Y)$ is an S_g-open set in X. But $f(x) \in Y$ for each $x \in X$, so $f^{-1}(U) = f^{-1}(U \cap Y)$ is an S_g-open subset of X. Therefore, by Proposition 2.2, $f: X \to Z$ is S_g-continuous.

Proposition 2.23 The following statements are equivalent for a function $f: X \to Y$:

1- f is RC-continuous.
2- f is S_g-continuous and contra-continuous.
3- f is semi-continuous and contra-continuous.
4- f is β-continuous and contra-continuous.
5- f is β-continuous and contra α-continuous.
6- f is β-continuous and contra-pre-continuous.

Proof. This is an immediate consequence of Proposition1.12.

Proposition 2.24 Let $f: X \to Y$ be an S_g-continuous function. If A is a clopen subset of X, then $f|A: A \to Y$ is S_g-continuous .

Proof. Let V be any open set of Y. Since f is S_g-continuous, then by Proposition 2.2, $f^{-1}(V)$ is an S_g-open set in X. Since A is a clopen subset of X, by Corollary 1.13, $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is an S_g-open subspace of A. This shows that $f|A: A \to Y$ is S_g-continuous.

Proposition 2.25 A function $f: X \to Y$ is S_g-continuous, if for each $x \in X$, there exists a regular closed set A of X containing x such that $f|A: A \to Y$ is S_g-continuous.

Proof. Let $x \in X$. Then by hypothesis, there exists a regular closed set A containing x such that $f|A: A \to Y$ is S_g-continuous, let V be any open set of Y containing $f(x)$. There exists an S_g-open set U in A containing x such that $(f|A)(U) \subseteq V$. Since A is a regular closed set. By Proposition 1.14, U is an S_g-open set in X, whence $f(U) = (f|A)(U) \subseteq V$. Thus f is S_g-continuous.

Proposition 2.26 If $X = A \cup B$, where A and B are regular closed sets and $f: X \to Y$ is a function such that both $f|A: A \to Y$ and $f|B: B \to Y$ are S_g-continuous function, then f is S_g-continuous.

Proof. Let V be any open set of Y. Then $f^{-1}(V) = (f|A)^{-1}(V) \cup (f|B)^{-1}(V)$. Since $f|A$ and $f|B$ are S_g-continuous. Then by Proposition2.2, $(f|A)^{-1}(V)$ and $(f|B)^{-1}(V)$ are S_g-open set in A and B, respectively. Since A and B are regular closed sets in X, then by Proposition1.14, $(f|A)^{-1}(V)$ and $(f|B)^{-1}(V)$ are S_g-open sets in X. Since union of two S_g-open sets is S_g-open. Hence $f^{-1}(V)$ is S_g-open in X. Therefore, by Proposition2.2, f is S_g-continuous.

Theorem 2.27 Let $f: X \to Y$ be a function and let $\{A_\lambda: \lambda \in \Delta\}$ be a regular closed cover of X. If the restriction $f|A_\lambda: A_\lambda \to Y$ is S_g-continuous for each $\lambda \in \Delta$, then f is S_g-continuous.

Proof. Let $f|A_\lambda: A_\lambda \to Y$ be S_g-continuous for each $\lambda \in \Delta$ and let U be any open subset in Y. Then by Proposition2.2, $(f|A_\lambda)^{-1}(U) \in S_gO(A_\lambda)$ for each $\lambda \in \Delta$. But $(f|A_\lambda)^{-1}(U) = f^{-1}(U) \cap A_\lambda \in S_gO(A_\lambda)$ for each $\lambda \in \Delta$. Since A_λ is regular closed for each $\lambda \in \Delta$, Then by Proposition1.14, $f^{-1}(U) \cap A_\lambda \in S_gO(X)$ for each $\lambda \in \Delta$. So $f^{-1}(U) = \cup_{\lambda\in\Delta}(f^{-1}(U) \cap A_\lambda) \in S_gO(X)$. Hence $f^{-1}(U) \in S_gO(X)$. Thus by Proposition2.2, f is S_g-continuous.

Proposition 2.28 If $f: X \to Y$ is an α-continuous function and X is locally indiscrete, then f is S_g-continuous.

Proof. Let $x \in X$ and let U be any open subset of Y containing $f(x)$. Since every α-open set is semi-open and X is locally indiscrete, then by Proposition1.6, f is S_g-continuous.

Theorem 2.29 Let $f: X \to Y$ be a surjection function. Then the following statements are equivalent:

1- f is S_g-continuous.

2- For every $B \subseteq Y$, $intclf^{-1}(B) \subseteq f^{-1}(clB)$ and $f^{-1}(clB) = \cap_{i\in\Delta} V_i$, where V_i is g-open in X.

3- For every $B \subseteq Y$, where $f^{-1}(intB) \subseteq clintf^{-1}(B)$ and $f^{-1}(intB) = \cup_{i\in\Delta} F_i$, where F_i is g-closed in X.

4- For every $A \subseteq X$, $f(intclA) \subseteq clf(A)$ and $f^{-1}(clf(A)) = \cap_{i\in\Delta} V_i$, where V_i is g-open in X.

Proof. (1) → (2) Let $B \subseteq Y$. Then clB is closed in Y. Since f is S_g-continuous, then by Proposition2.20, $f^{-1}(clB)$ is S_g-closed in X. Therefore, by Proposition1.3, $f^{-1}(\text{cl}B)$ is semi-closed and $f^{-1}(clB) = \cap_{i\in\Delta} V_i$, where V_i is g-open in X. Thus $intclf^{-1}(clB) \subseteq f^{-1}(clB)$ and $f^{-1}(clB) = \cap_{i\in\Delta} V_i$, where V_i is g-open in X.

(2) → (3) Let $B \subseteq Y$. Then $Y\backslash B \subseteq Y$. So by (2), $intclf^{-1}(Y\backslash B) \subseteq f^{-1}(cl(Y\backslash B))$ and $f^{-1}(Y\backslash B) = \cap_{i\in\Delta} V_i$, where V_i is g-open for each $i \in \Delta$. Then $X\backslash clintf^{-1}(B) \subseteq X\backslash f^{-1}(intB)$ and $X\backslash f^{-1}(intB) = \cap_{i\in\Delta} V_i$, where V_i is g-open for each $i \in \Delta$. Then $f^{-1}(intB) \subseteq clintf^{-1}(B)$ and $f^{-1}(intB) = \cup_{i\in\Delta}(X\backslash V_i)$, where $X\backslash V_i$ is g-closed for each $i \in \Delta$.

(3) → (1) Let B be open subset of Y. Then $intB = B$, so by (3) $f^{-1}(B) \subseteq clintf^{-1}(B)$ and $f^{-1}(B) = \cup_{i\in\Delta} F_i$, where F_i is g-closed in X. Thus $f^{-1}(B) \in S_gO(X)$. Hence f is an S_g-continuous function.

(2) → (4) Let $A \subseteq X$. Then $f(A) \subseteq Y$, so by (2), $intclf^{-1}(f(A)) \subseteq f^{-1}(clf(A))$ and $f^{-1}(clf(A)) = \cap_{i\in\Delta} V_i$, where V_i is g-open in X. Therefore, $intclA \subseteq f^{-1}(clf(A))$ and $f^{-1}(clf(A)) = \cap_{i\in\Delta} V_i$, where V_i is g-open in X. Hence $f(intclA) \subseteq clf(A)$ and $f^{-1}(clf(A)) = \cap_{i\in\Delta} V_i$, where V_i is an g-open set in X.

(4) → (2) Let $B \subseteq Y$. Then $f^{-1}(B) \subseteq X$. Therefore, by (4) $f\ (intclf^{-1}(B)) \subseteq clf(f^{-1}(B)) \subseteq clB$ and $f^{-1}(clB) = \cap_{i\in\Delta} V_i$, where V_i is g-open in X. This implies that, $intclf^{-1}(B) \subseteq f^{-1}(clB)$ and $f^{-1}(clB) = \cap_{i\in\Delta} V_i$, where V_i is an g-open set in X.

The composition of two S_g-continuous functions need not be S_g-continuous, as it is shown by the following example:

Example 2.30 Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$ and $\sigma = \{\emptyset, X, \{c\}, \{b\}, \{c, b\}\}$, then $S_gO(X, \tau) = \{\emptyset, X, \{a, c\}\}$ and $S_gO(X, \sigma) = \{\emptyset, X, \{a, c\}, \{a, b\}\}$. Let $f: (X, \tau) \to (X, \sigma)$ be a function defined by: $f(a) = f(c) = b$ and $f(b) = a$. Then f is S_g-continuous, let $g: (X, \sigma) \to (X, \tau)$ be a function defined by: $g(a) = g(c) = a$ and $g(b) = b$. Then g is S_g-continuous but $gof: (X, \tau) \to (X, \tau)$ is not S_g-continuous. Since $gof(b) = \{a\} \in \tau$ but $(gof)^{-1}(a) = \{b\} \notin S_gO(X, \tau)$.

Theorem 2.31 Let $f: X \to Y$ be an S_g-continuous function and $g: Y \to Z$ be a continuous function. Then $gof: X \to Z$ is S_g-continuous.

Proof. Let W be any open subset of Z. Since g is a continuous function, then $g^{-1}(W)$ is open subset of Y. Since f is S_g-continuous, then by Proposition2.2, $(gof)^{-1}(W) = f^{-1}(g^{-1}(W))$ is an S_g-open subset in X. Thus by Proposition2.2, gof is S_g-continuous.

Proposition 2.32 Let X, Y_1 and Y_2 be three spaces and $f_i: X \to Y_i$ for $i = 1, 2$ are functions. If the function $g: X \to Y_1 \times Y_2$ defined as, $g(x) = (x_1, x_2)$, where $f_i(x) = x_i$ for $i = 1, 2$ is S_g-continuous, then f_i is S_g-continuous for $i = 1, 2$.

Proof. Let $x \in X$ and V_1 be any open set in Y_1 containing $f_1(x) = x_1$. Then $V_1 \times Y_2$ is open in $Y_1 \times Y_2$ which contain (x_1, x_2). Since g is S_g-continuous. Then by Proposition 2.2, $g^{-1}(V_1 \times Y_2)$ is S_g-open in X. But $f^{-1}(V_1) = g^{-1}(V_1 \times Y_2)$. Thus f_1 is S_g-continuous. Similarly we can prove that f_2 is S_g-continuous.

Proposition 2.33 Let $f, g: X \rightarrow Y$ be functions and Y is Hausdorff. If f is S_g-continuous and g is clopen continuous, then the set $E = \{x \in X: f(x) = g(x)\}$ is S_g-closed in X.

Proof. Let $x \notin E$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, then there exist open sets V_1 and V_2 of Y such that $f(x) \subseteq V_1$, $g(x) \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. Since f is S_g-continuous, there exists an S_g-open subset U_1 of X containing x such that $f(U_1) \subseteq V_1$. Since g is clopen continuous there exists a clopen subset U_2 of X containing x such that $g(U_2) \subseteq V_2$. Then by Proposition1.15, the set $U = U_1 \cap U_2$ is an S_g-open set of X containing x, and $U \cap E = \emptyset$. Therefore, we obtain $x \notin S_g cl(E)$. This is show that E is S_g–closed in X.

Proposition 2.34 Let $f, g: X \rightarrow Y$ be S_g-continuous functions such that $S_g O(X)$ is a topology on X and Y is Hausdorff. Then the set $E = \{x \in X: f(x) = g(x)\}$ is S_g-closed in X.

Proof. Let $x \notin E$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, then there exist open sets V_1 and V_2 of Y such that $f(x) \subseteq V_1$, $g(x) \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. Since f and g are S_g-continuous, then there exist S_g-open sets U_1 and U_2 in X containing x such that $f(U_1) \subseteq V_1$ and $g(U_2) \subseteq V_2$. Then by hypothesis, the set $W = U_1 \cap U_2$ is an S_g-open set of X containing x such that $U \cap E = \emptyset$. Therefore, we obtain $x \notin S_g cl(E)$. This is show that E is S_g–closed in X.

Proposition 2.35 Let $f: X_1 \rightarrow Y$ and $g: X_2 \rightarrow Y$ be two S_g-continuous functions. If Y is a Hausdorff space, then the set $E = \{(x_1, x_2) \in X_1 \times X_2: f(x_1) = g(x_2)\}$ is S_g-closed in the product space $X_1 \times X_2$.

Proof. Let $(x_1, x_2) \notin E$. Then $f(x_1) \neq g(x_2)$. Since Y is Hausdorff, there exist open sets $V_1$and V_2 of Y such that $f(x_1) \subseteq V_1$, $g(x_2) \subseteq V_2$, and $V_1 \cap V_2 = \emptyset$. Since f and g are S_g-continuous there exist S_g-open sets U_1 and U_2 of X_1 and X_2 containing x_1 and x_2 such that $f(U_1) \subseteq V_1$ and $g(U_2) \subseteq V_2$, respectively. Then by Proposition 1.16, the set $U = U_1 \times U_2$ is an S_g-open subset of $X_1 \times X_2$ containing (x_1, x_2) such that $U \cap E = \emptyset$. Therefore, we obtain $(x_1, x_2) \notin S_g cl(E)$. Hence E is an S_g-closed set in the product space $X_1 \times X_2$.

Corollary 2.36 Let $f: X \rightarrow Y$ be S_g-continuous function and X is a $T_{\frac{1}{2}}$ –space.Then:

1- f is S_s-continuous.
2- f is S_β-continuous.
3- f is S_p-continuous.

Proof. Follows directly by Corollary1.17.

References

[1] Abd-El-Monsef M. E., El-Deeb S. N. and Mahmoud R. A. , "β-open sets and β-continuous mappings", Bull. Fac. Assint, Vol. 12, 77-90, (1983).

[2] Darwesh H. M., "A new Type of Semi-open sets and Semi- Continuity in Topological Spaces", Zanko J. of Pure and Appl. Sci., Vol. 23, No.2, 82-92, (2011).

[3] Darwesh H. M. and Hessean N. O., " S_g-open sets in topological spaces", J. of Zankoy Sulaimani for Pure and Appl. Sci., in press (2015).

[4] Dontchev J., "Survey on preopen sets", ArXvi Math., Vol. 1, 1-18, (1998).

[5] Dontchev J., "Contra-continuous functions and strongly S-closed spaces", Int. J. Math. and Math. Sci.,
Vol. 19, No. 2, 303-310, (1996).

[6] Dontchev J. and Noiri T., "Contra-semi continuous functions", Math. Pannon, Vol. 10, No. 2, 159-168, (1999).

[7] Guo T. Y., "A characterization of extermally disconnected spaces", J. central china Normal Univ. Nature. Sci., Vol. 12, No. 2, 169-170, (1981).

[8] Jafari S. and Noiri T., "On contra-precontinuous functions", Bull. Malays. Math. Sci., Vol. 25, No. 2, 115-128, (2002).
[9] Jafari S. and Noiri T., "Contra α-continuous functions between topological spaces", Iran. Int. J. Sci., Vol. 2, No. 2, 153-167, (2001).
[10] Jain R. C., "The role of regularly open sets in general topology", Ph.D. Thesis, Meerut Univ. Institute of advanced studies, Meerut-India, (1980).
[11] Joseph J. E. and Kwack M. H., " On S-closed spaces", Proc. Amer. Math. Soc., Vol. 80, No. 2, 341-348, (1980).
[12] Khalaf A. B. and Ahmed N. K., " S_β-open sets and S_β-continuity in topological spaces", THAI J. of Math. Monthly, Vol. 70, No. 1, 319-335, (2013).
[13] Khalaf A. B. and Aameen Z. A., " S_c-open sets and S_c-continuity in topological spaces", J. of Advanced Research in Pure Math. Vol. 2, No. 3, 87-101, (2010).
[14] Khalaf A. B. and Easif F. H., " θs-continuous functions", J. Dohuk Univ., special issue, 1-4, (1999).
[15] Khalaf A. B., Majeed A. H. and Jamil J. M., " S_s-open sets and S_s-continuous Functions", Int. J. of Advanced Math. Sci., Vol. 2, No. 1, 71-80, (2014).
[16] Levin N., "Generalized closed sets in topology", Rend. Circ. Math. Paleremo Vol. 19, No. 2, 89-96, (1970).
[17] Levine N., "Semi-open sets and semi-continuity in topological spaces", Amer. Math. Monthly, Vol. 70, 36-41, (1963).
[18] Majid M., " A note on extermally disconnected spaces", Research Open J. of Information Sci., Vol. 1, No. 1, 01-03, (2013).
[19] Mashhour A. S., Abd El-Monsef M. E. and El-Deeb S. N. "On precontinuous and weak precontinuous functions", Proc. Math. Phy. Soc. Egypt, Vol. 53, 47-53, (1982).
[20] Mashhour A. S., Hasanein I. A. and El-Deeb S. N., "A note on semi-continuity and precontinuity", Indian J. Pure Appl. Math. Vol. 13, No. 10, 1119-1123, (1982).
[21] Mashhour A. S., Hasanein I. A. and El-Deeb S. N., " α-continuous and α-open mapping", Acta Math. Hung., Vol. 41, No. (3-4), 213-218, (1983).
[22] MOIZ-UD-DIN KHAN, "Weak forms of continuity, compactness and connectedness", Ph.D. Thesis, Bahauddin Zakariya Univ. Multan, Pakistan, (1997).
[23] Munshi B. M. and Bassan D. S., "Super continuous functions", Indian J. Pure Appl. Math., Vol. 13, No. 2, 229-236, (1982).
[24] Njastad O., "On some classes of nearly open sets", Pacific J. Math., Vol. 15, No. 3, 961-970, (1965).
[25] Noiri T. , "Super continuity and some strong forms of continuity", Indian J. Pure Appl. Math., Vol. 15, No. 3, 241-250, (1984).
[26] Noiri T. and Popa V., "Weak forms of faint continuity", Bull. Math. de la Soc. Sci. Math. de la Roumanie, Vol. 34,No. 82, 263-270, (1990).
[27] Nour T. M., "Totally semi-continuous functions", Indian J. Pure Appl. Math., Vol. 26, No. 7, 675-678, (1995).
[28] Park J. H., Lee B. Y., and Son M. J., "On δ-semi-open sets in topological spaces", J. Indian Acad. Math. Vol. 19, No. 1, 59-67, (1997).
[29] Reilly I. L. and Vamanmurthy, "On super-continuous functions", Indian J. Pure Appl. Math., Vol. 14, 767-772, (1983).

[30] Shareef H. A., " S_p -open sets, S_p-continuity and S_p-compactness in Topological spaces", M.Sc. Thesis, Univ. of Sulaimani, (2007).

[31] Simmons G. F., "Topology and Modern Analysis", Tata McGraw Hill, (1963).

[32] Steen L. A. and Seebach J. A., "Counterexamples in Topology, Springer Verlag New York Heidelberg Berlin, (1978).

[33] Stone M. H., "Algebraic characterizations of special Boolean rings", Fund. Math., Vol. 29, 223-302, (1937).

[34] Velicko N. V. , "H-closed topological spaces", Amer. Math. Soc. Transl., (1968). Vol. 78, No. 2, 103-118.

[35] Willard S., "General Topology", Addison-Wesley Publishing Company, (1970).

[36] Yunis R. H., "Properties of θ-semi-open sets", Zanco J. of Pure and Appl. Sci., Vol. 19, No. 1, 116-122, (2007).